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The Zelmanov Nilpotence Theorem for Quadratic Jordan Algebras

KEVIN MCCRIMMON*

Department of Mathematics, University of Virginia, Charlottesville, Virginia 22901

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TO NATHAN JACOBSON ON HIS 70TH BIRTHDAY

In the Jacobson theory of Jordan rings with descending chain condition on inner ideals, the only gap was the proof of nilpotency of the Jacobson radical. E. I. Zelmanov has recently established this by analyzing the locally nilpotent radical. Although his result is stated for linear Jordan algebras, we show that the proof can be extended to arbitrary quadratic Jordan rings.

Throughout we consider unital quadratic Jordan algebras J over an arbitrary ring of scalars Φ. J is thus a unital Φ-module with unit 1 and product $U(x)y$ quadratic in x and linear in y, satisfying the axioms

$$U(1) = \mathrm{Id}, \qquad U(x)\,V(y, x) = V(x, y)\,U(x), \qquad U(U(x)y) = U(x)\,U(y)\,U(x)$$

for

$$V(x, y)z = \{x\,y\,z\} = [U(x+z) - U(x) - U(z)]y = U(x, z)y,$$
$$V(x) = V(x, 1).$$

The unit element determines squaring and circling operations

$$x^2 = U(x)1, \qquad x \circ y = U(x, y)1 = V(x)y.$$

The archetypal example of a quadratic Jordan algebra is the structure A^+ obtained from an associative algebra A by defining

$$U(x)y = xyx, \qquad V(x, y)z = xyz + zyx, \qquad x^2 = xx, \qquad x \circ y = xy + yx.$$

The Jordan subalgebras of algebras A^+ are called *special* Jordan algebras; examples are the Hermitian elements $H(A, *)$ when $*$ is an involution on A, or

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the subspace $J(Q, c) = X$ of the Clifford algebra $C(Q, c) = A$ of a quadratic for mon X with basepoint c.

A subspace $B \subset J$ is an *inner* (or *quadratic*) *ideal* if it is closed under inner multiplication by J, $U(B)J \subset B$, similarly an *outer ideal* is closed under outer multiplication, $U(J)B \subset B$, while an *ideal* is inner and outer. The inner ideals play the role in Jordan theory that one-sided ideals do in associative theory. Jacobson's recognition [2] of the fundamental importance of inner ideals led directly to the structure theory and the recasting of Jordan algebras as a quadratic theory [6, 4].

The structure theory was formulated for *nondegenerate* algebras, those having no *trivial elements* (or *absolute zero divisors*) z such that $U(z) = 0$. The *degenerate radical* $L(J)$ is the smallest ideal whose quotient is nondegenerate. For example, the degenerate radical of $J(Q, c)$ is just the radical of the quadratic form Q, and $J(Q, c)$ is nondegenerate iff Q is. Jacobson recognized the importance of trivial elements: they are the useful notion of "bad" elements, and nondegeneracy the practical form of "well-behavedness." One wished to know that this practical definition could be theoretically justified, and the nondegenerate radical related to the standard ones.

In the presence of the descending chain condition on inner ideals, nondegeneracy coincides with *semisimplicity* [7], the vanishing of the *Jacobson radical* (the maximal ideal which is *radical* in the sense that it consists entirely of quasi-invertible elements z, those for which $1 - z$ is invertible). However, one really wanted to know that nondegeneracy coincided with *semiprimeness*, the nonexistence of nilpotent ideals. As usual, an ideal is *nilpotent* of index n if all products with n or more factors vanish (where we count $U(x)y$ as having two factors x). This coincidence was known for finite-dimensional algebras [8], but the general case of algebras with d.c.c. long remained open. It was finally settled for algebras over fields of characteristic $\neq 2$ by Slin'ko and Zelmanov [15] and over arbitrary rings containing $\frac{1}{2}$ by Zelmanov. Actually, Zelmanov proved [12, 13] a much more general result: if $\frac{1}{2} \in \Phi$ and J has either d.c.c. or a.c.c. on inner ideals, then the nil radical of J is nilpotent. In this paper we will restrict ourselves to the case of algebras with d.c.c. Our task is to extend the result to arbitrary quadratic Jordan algebras.

For convenience we collect here for future reference an intimidating list of multiplication identities valid in any Jordan algebra. (Familiarity with these is not a prerequisite for reading the rest of this article!)

$$U(U(x)y) = U(x)\, U(y)\, U(x), \qquad U(x^2) = U(x)\, U(x), \tag{0.1}$$

$$V(x, y)\, U(x) = U(x)\, V(y, x), \qquad V(x)\, U(x) = U(x)\, V(x), \tag{0.2}$$

$$V(U(x)y, y) = V(x, U(y)x), \qquad V(U(x)y, z) + V(U(x)z, y) = V(x, \{yxz\}), \tag{0.3}$$

$$V(x^2, z) + V(U(x)z) = V(x, x \circ z), \qquad V(x^2) = V(x, x),$$
$$V(y \circ z) = V(y, z) + V(z, y), \tag{0.4}$$

$$V(x, y) = V(x) V(y) - U(x, y), \qquad V(x) = U(x, 1),$$
$$2U(x) = V(x)^2 - V(x^2), \tag{0.5}$$

$$V(x, y) U(z) + U(z) V(y, x) = U(\{xyz\}, z), \tag{0.6}$$

$$V(x) U(z) + U(z) V(x) = U(x \circ z, z), \tag{0.7}$$

$$U(U(x)y, z) = U(x, z) V(y, x) - V(z, y) U(x) \tag{0.8}$$
$$= V(x, y) U(x, z) - U(x) V(y, z),$$

$$V(U(x)y, z) = V(x, y) V(x, z) - U(x) U(y, z), \tag{0.9}$$
$$V(z, U(x)y) = V(z, x) V(y, x) - U(y, z) U(x),$$

$$V(U(x)y) = V(x, y) V(x) - U(x) V(y)$$
$$= V(x) V(y) V(x) - U(x, y) V(x) - U(x) V(y), \tag{0.10}$$

$$U(\{xyz\}) = U(x) U(y) U(z) + U(z) U(y) U(x)$$
$$+ V(x, y) U(z) V(y, x) - U(U(x) U(y)z, z), \tag{0.11}$$

$$U(x \circ z) = U(x) U(z) + U(z) U(x) + V(x) U(z) V(x) - U(U(x)z, z)$$
$$= U(x) U(z) + U(z) U(x) + V(x) U(z) V(x)$$
$$- V(x, z) U(x, z) + U(x) V(z, z), \tag{0.12}$$

$$[U(x), U(y)] = U(x \circ y) + U(y) V(x^2) + U(U(x)y, y) - U(x \circ y, y) V(x). \tag{0.13}$$

The reader may check these for himself using Macdonald's Theorem, or refer to our standard references for results on Jordan algebras [4, p. 1.16–1.22, 5; p. 13–20; 3], or accept them on faith.

1. Motivation

Let us recall, for motivation and guidance, the proof of nilpotency of the Jacobson radical R of an associative algebra A with d.c.c. on left ideals. (1) First we reduce to an idempotent radical ideal: by the d.c.c. on two-sided ideals we have $R \supset R^2 \supset \cdots \supset R^n = R^{n+1} = \cdots = S$ for some n. If $S = 0$ then R is nilpotent and we are done, so assume $S \neq 0$. Then $S^2 = R^{2n} = R^n = S$ is an idempotent ideal inside R. (2) Next we find a left ideal I minimal among those not annihilated by S, and show $I = Sx$ for some element $x \in I$: if $SI \neq 0$ then $Sx \neq 0$ for some $x \in I$, hence the left ideal $I' = Sx \subset I$ has $SI' = SSx = Sx \neq 0$ by idempotence of S, so by minimality of I we must have $I = I' = Sx$.

(3) We reach a contradiction, since $x \in I$ but $x \notin Sx$ for radical S: $x = sx$ would imply $(1 - s)x = 0$ where $1 - s$ is invertible since $s \in S \subset R$ is quasi-invertible, therefore $x = 0$, a contradiction.

We will try to repeat this procedure in the Jordan case. Here there are major differences between three nilpotence concepts for an ideal: *nilpotence*, *Penico-solvability* (eventual vanishing of the *Penico derived series* $P^n(R)$, for $P(R) = U(R)J$, $P^{n+1}(R) = P(P^n(R))$), and *solvability* (eventual vanishing of the *derived series* $D^n(R)$, for $D(R) = U(R)R$ and $D^{n+1}(R) = D(D^n(R))$). Note these derived series consist of ideals in J because of the general principle

$$B, C \lhd J \Rightarrow U(B)C \lhd J.$$

The associative outline we are following leads naturally in Section 4 from local nilpotence of the radical R to solvability of R, so we must backtrack in Section 5 to establish local nilpotence (the hardest part), and then afterward establish Penico solvability in Section 6 and nilpotence in Section 7.

2. The Idempotent Ideal S

So let J be a unital Jordan algebra with radical ideal R (e.g., $R = \mathrm{Rad}(J)$ the Jacobson radical), and at first assume only the d.c.c. on *ideals* contained in R. Then the derived series $R \supset D(R) \supset D^2(R) \supset \cdots \supset D^n(R) = D^{n+1}(R) = \cdots$ eventually terminates at an idempotent ideal $S : D(S) = D(D^n(R)) = D^{n+1}(R) = S$. We need not merely the idempotence of S, but also idempotence of multiplication by S, i.e., of its associative multiplication algebra.

2.1. Lemma. *The multiplication algebra $M_J(S) \subset \mathrm{End}(J)$ generated by the operators $U(S)$, $V(S)$ for an idempotent ideal S in J is an idempotent associative algebra*: $M_J(S)^2 = M_J(S)$.

Proof. $S = U(S)S$ is spanned by elements $U(x)y$ for x, y in S, so $U(S)$, $V(S)$ are spanned by operators $U(U(x)y)$, $U(U(x)y, z)$, $V(U(x)y)$, and these generators lie in $M_J(S)^2$ by (0.1), (0.8), (0.10). ∎

The multiplication ideal $M_J(S, J)$ generated by $U(S)$ and $V(S, J)$ is similarly idempotent, but unfortunately it does not seem to be the case that the algebra $U_J(S)$ generated by $U(S)$, or the ideal $W_J(S) \lhd M_J(S)$ generated by $U(S)$, is idempotent when S is.

3. The Minimal Inner Ideal I

Now assume J has d.c.c. on inner ideals contained inside an idempotent radical ideal S. The naive approach based on the associative example fails. Namely, if we use the d.c.c. to find an inner I minimal with respect to $U(S)I \neq 0$

(or perhaps we could try $U(I)S \neq 0$), we choose $x \in I$ with $U(S)x \neq 0$ (respectively $U(x)S \neq 0$), so the usual principal inner ideal $I' = U(x)J \subset I$ will have $x \notin I'$ (because $x = U(x)a$ for quasi-invertible $x \in S \subset R$ implies $x = 0$), and therefore $I' < I$, but we do not contradict the minimality of I because we do not necessarily have $U(S)I' = U(S)\,U(x)J$ nonzero just because $U(S)x$ is nonzero (or $U(I')S = U(x)\,U(J)\,U(x)S \neq 0$ just because $U(x)S \neq 0$). The trouble seems to be that these new relations involve more x's than the original ones, because our principal inner ideal $U(x)J$ is quadratic instead of linear in x.

This suggests we try to find a different inner ideal generated by S and x, one which is linear in x in analogy to Sx of the associative case. A natural candidate is $I' = M_J(S)x$ ($U(S)x$ is not enough since $U_J(S)$ may not be idempotent). When $\frac{1}{2} \notin \Phi$ we must include some complicated quadratic terms to make this an inner ideal. Notice also that if we want $I' = M_J(S)x$ to stay inside I, we should take I to be an ideal of S instead of merely an inner ideal of J. For convenience, we write $\hat{M}_J(S) = \Phi 1 + M_J(S)$ for the result of adjoining the identity operator to $M_J(S)$.

3.1. Construction Lemma. *Let S be an ideal in a Jordan algebra J, x an element of S. Then*

$$I(S, x) = M_J(S)x + \hat{M}_J(S)\,U(x)S + \hat{M}_J(S)\,U(S)\,U(x)J \mathrel{\triangleleft\!\!\!\cdot} J, \triangleleft\, S \quad (3.2)$$

is an inner ideal of J and an ideal of S, while

$$I_0(S, x) = M_J(S)x + \hat{M}_J(S)\,U(x)\hat{S} \triangleleft S \quad (3.3)$$

is an ideal of S, and

$$U(y)J \subset I_0(S, x) \qquad \text{for all} \quad y \in \hat{M}_J(S)\,U(x)S + \hat{M}_J(S)\,U(S)\,U(x)J. \quad (3.4)$$

When $\frac{1}{2} \in \Phi$ these simplify to $M_J(S)x$:

$$I(S, x) = I_0(S, x) = M_J(S)x \qquad (\tfrac{1}{2} \in \Phi).$$

Proof. In proving innerness of $I(S, x)$ and the pushing property (3.4), it will be convenient to formalize the common induction in both cases, namely, that if $U(y)J \subset K$ for certain elements y then the same holds for the S-ideal they generate.

3.5. Sublemma. *If $S \triangleleft J$ is an ideal in J and K an S-invariant subspace, $M_J(S)K \subset K$, then*

$$T(S, K) = \{y \in U(S)\hat{S} \mid U(y)J + V(S)y + V(S, S)y \subset K\} \triangleleft S, \mathrel{\triangleleft\!\!\!\cdot} J$$

is an ideal in S and an inner ideal in J.

Proof. To show T is a subspace we make use of our assumption that y lies in $U(S)\hat{S} = U(S)S + S^2$ instead of merely in S. Indeed, the nonlinear condition is satisfied by $y + y'$ when y, y' lie in T since $U(y + y') = U(y) + U(y') + U(y, y')$, where

$$U(y, y')J = V(y', J)y \subset V(U(S)\hat{S}, J)y \subset \{V(S) + V(S, S)\}y \subset K$$

by hypothesis, noting

$$\begin{aligned} V(U(S)S, J) &\subset V(S, S), \\ V(S^2, J) &\subset V(S) + V(S, S) \end{aligned} \tag{3.6}$$

by linearized (0.3), (0.4).

To see T is inner in J, $y' = U(y)a \in T$ if $y \in T$ and $a \in J$, we observe that $y' \in K$ so $\{V(S) + V(S, S)\}y' \subset K$ by S-invariance of K, and $U(y')J = U(y)\, U(a)\, U(y)J \subset U(y)J \subset K$, and finally y' belongs to $U(S)\hat{S}$ (this is an inner ideal in J since $U(S)S$ is an ideal in J and $U(s^2)J = U(s)\, U(s)J \subset U(S)S$, $U(s^2, t^2)J = \{V(s)\, U(s, t^2) - U(s)\, V(t^2)\}J$ (by (0.8) with $y = 1) \subset S \circ S + U(S)S \subset U(S)\hat{S}$).

In particular, T is inner in S. To see it is also outer, $y' = U(s')y \in T$ if $y \in T$, $s' \in \hat{S}$, note y' remains in $U(S)\hat{S}$ and has $U(y')J = U(s')\, U(y)\, U(s')J \subset U(\hat{S})\, U(y)J \subset U(\hat{S})K \subset K$, while $\{V(S) + V(S, S)\}\, y' \subset \{V(S) + V(S, S)\}\, \{I + V(S) + U(S)\}y \subset \hat{M}_J(S)\{V(S) + V(S, S)\}y \subset \hat{M}_J(S)K \subset K$, from

$$\{V(S) + V(S, S)\}\, \hat{M}_J(S) = \hat{M}_J(S)\{V(S) + V(S, S)\} \tag{3.7}$$

(using (0.5), linearized (0.7) to move past $V(S)$ and (0.6), (0.9) to move past $U(S)$). Thus T is an ideal in S. ∎

We actually only need the outerness of T. We first apply the sublemma with $K = I(S, x) = I$ to prove I is an inner ideal in J and an ideal in S as in (3.2). By construction I is an outer ideal in S, so it suffices to prove it is an inner ideal in J, $U(I)J \subset I$, thus to prove $I \subset T(S, I)$ (since the other conditions $I \subset U(S)\hat{S}$ and $\{V(S) + V(S, S)\}I \subset I$ of 3.5 are automatically met—note $V(S)x \subset S^2$ but $\not\subset U(S)S$, which is why we had to use $U(S)\hat{S}$ in the sublemma). But here it suffices to prove the S-generators $y = V(s)x$, $U(s)x$, $U(x)s$, $U(s)\, U(x)a$ of I fall in T, and these boost J into I ($U(y)J \subset I$) since, by (0.1), (0.12),

$$\begin{aligned} U(U(s)x)J &= U(s)\, U(x)\, U(s)J \subset M(S)\, U(x)S \subset I_0, \\ U(U(x)s)J &= U(x)\, U(s)\, U(x)J \subset U(x)S \subset I_0, \\ U(U(s)\, U(x)a)J &= U(s)\, U(x)\, U(a)\, U(x)\, U(s)J \subset M(S)\, U(x)S \subset I_0, \\ U(V(s)x)J &= \{U(s)\, U(x) + U(x)\, U(s) + V(x)\, U(s)\, V(x) \\ &\qquad - V(x, s)\, U(x, s) + U(x)\, V(s, s)\}J \\ &\subset U(S)\, U(x)J + U(x)S + V(x)S - V(x, S)S + U(x)S \\ &\subset U(S)\, U(x)J + U(x)S + V(S)x + V(S, S)x \subset I \end{aligned} \tag{3.8}$$

(note here the generator $y = V(s)x$ forces I to contain $U(S)\,U(x)J$, otherwise we could be content with I_0). Thus $I(S, x) \lhd S, \lhd J$.

Applying this to the unital Jordan algebra $\hat{S}$ instead of J, $I(S, x)$ becomes $I_0(S, x)$ since $\hat{M}_J(S)\,U(S)\,U(x)J$ becomes $\hat{M}_J(S)\,U(S)\,U(x)\hat{S}$. Thus $I_0(S, x)$ is an ideal in S.

For the pushing assertion (3.4) it suffices to prove $\hat{M}_J(S)\,U(x)S + \hat{M}_J(S)\,U(S)\,U(x)J$ belongs to $T(S, K)$ for $K = I_0(S, x)$, and once again by the sublemma it is enough if this holds for the S-generators $y = U(x)s$, $U(s)\,U(x)a$. Here y belongs to $U(S)S$ since $x \in S$, satisfies $U(y)J \subset I_0$ by (3.8), and satisfies $\{V(S) + V(S, S)\}y \subset I_0$ since $y = U(x)s$ is already in I_0 which is S-invariant, and $y = U(s)\,U(x)a$ is boosted into I_0 by any linear S-multiplication since $\{V(S) + V(S, S)\}y \subset \hat{M}_J(S)\{V(S) + V(S, S)\}\,U(x)a$ (by (3.7)), where by linearized (0.2) we have $V(S)\,U(x)J \subset \{-V(x)\,U(x, s) + U(x, s)\,V(x) + U(x)\,V(S)\}J \subset V(x)S + U(x, S)S + U(x)S = V(S)x + V(S, S)x + U(x)S \subset I_0$, and similarly $V(S, S)\,U(x)J \subset V(x, S)S + U(x, S)S + U(x)S \subset I_0$. This establishes (3.4).

The quadratic terms disappear when multiplied by 2: $2U(S)\,U(x)J = U(S, S)\,U(x)J \subset I_0$ by the above, and $2U(x)S = U(x, x)S = V(x, S)x \subset V(S, S)x$. Thus $4I \subset 2I_0 \subset M_J(S)x$, so $I = I_0 = M_J(S)x$ when we can divide by 2. ∎

Having found the annoyingly complicated inner ideal $I(S, x)$, we can resume our associative analogy. By the d.c.c. we can choose I minimal with respect to the properties (i) $I \lhd J$ is an inner ideal in J, (ii) $I \lhd S$ is an ideal in S, (iii) $M_J(S)I \neq 0$. (At worst we could take S itself, since $M_J(S)S \supset U(S)S \neq 0$.) The hypothesis (iii) guarantees that $M_J(S)x \neq 0$ for some $x \in I$. The hypotheses (i), (ii) that I is an ideal in S as well as an inner ideal in J then guarantee $U(x)J \subset I$ and $I(S, x) \subset I$. Moreover, $M_J(S)\,I(S, x) \supset M_J(S)\,M_J(S)x = M_J(S)x \neq 0$ by idempotence (2.1) of $M_J(S)$. Therefore by minimality of I we have

$$I = I(S, x) = M_J(S)x + \hat{M}_J(S)\,U(x)S + \hat{M}_J(S)\,U(S)\,U(x)J. \tag{3.9}$$

In particular, $x \in I(S, x)$. Our next step is to show this leads to a contradiction when S is locally nilpotent.

3.10. *Remark.* The above construction of an inner ideal can be generalized from a single element x to an arbitrary subspace $B \subset S$

$$I(S, B) = M_J(S)B + \hat{M}_J(S)\,U(B)S + \hat{M}_J(S)\,U(S)\,U(B)J \lhd S, \lhd J.$$

If B is an inner ideal in J or an ideal in S this formula simplifies,

$$I(S, B) = M_J(S)B + U(B)S \qquad (\text{if } B \lhd J),$$
$$I(S, B) = S \circ B + U(S)B + U(B)S + \hat{M}_J(S)\,U(S)\,U(B)J \qquad (\text{if } B \lhd S),$$
$$I(S, B) = S \circ B + U(S)B + U(B)S \qquad (\text{if } B \lhd J,\ B \lhd S).$$

When $\frac{1}{2} \in \Phi$ these simplify even further:

$$I(S, B) = S \circ B \lhd J, \lhd S \qquad \text{if} \quad \tfrac{1}{2} \in \Phi, B \lhd S. \tag{3.11}$$

This was the construction used by Slin'ko [15, Lemma 1, p. 381] in the case of linear Jordan algebras. The important point is that $S \circ B$ is automatically inner in J even if B was not to begin with.

These formulas for the "S-ideal, J-inner ideal" generated by $M(S)x$ or B should be compared with the formula for the J-inner ideal generated by B:

$$\operatorname{Inn}(B) = B + \sum V(B, J)^n U(B) J \lhd J. \quad \blacksquare \tag{3.12}$$

4. Locally Nilpotent Implies Solvable

Our choice of minimal I has led to $x \in I = I(S, x)$, so $x = M(x)$ for some complicated multiplication as in (3.9)

$$x = M_0 x + \sum M_i U(x) s_i + \sum M_j U(s_j)\, U(x) a_j \qquad (s_i \in S, a_j \in J), \tag{4.1}$$

where $M_0 \in M_J(S)$ and the $M_k \in \hat{M}_J(S)$. Since this no longer has the form of a simple multiplication $x = sx$ or $x = U(x)s$, we cannot directly deduce a contradiction $x = 0$ from quasi-invertibility of S. Indeed, x is so deeply buried in the above product that it is not clear how quasi-invertibility of S can help us. In this way the quadratic nature of Jordan products leads naturally to the question of *local nilpotence* of the radical (i.e., that all finitely generated subalgebras are nilpotent), recalling the familiar lemma of Slater and Zhevlakov

$$x \in I_0(S, x) = M(S)x + \hat{M}(S)\, U(x)\hat{S} \Rightarrow x = 0. \tag{4.2}$$

(as in [8, p. 475], if $x = M_0 x + \sum M_i U(x)\, s_i + M_n U(x) 1$ for M_0, M_i involving a finite number of elements $t_1, \ldots, t_m$ from S then $S_0 = \Phi[x, t_1, \ldots, t_m, s_1, \ldots, s_n]$ is nilpotent by local nilpotence of S, where $x \in S_0{}^1$ and if $x \in S_0{}^k$ the above expression forces $x \in S_0^{k+1}$, therefore eventually $x \in S_0{}^N = 0$). However, because of our desire to make I an inner ideal in J we added the terms $U(S)\, U(x) J$ to I_0. Because of the presence of the a_j from J in (4.1) that relation $x = M(x)$ does not take place inside a finitely generated subalgebra of S where we could apply local nilpotence. We must first show $x = M(x) \in I(S, x)$ forces $x = M_0(x) \in I_0(S, x)$, and then we can apply (4.2).

4.3. Generalized Slater–Zhevlakov Lemma. *If S is a locally nilpotent ideal of a Jordan algebra J then a nonzero element x of S cannot lie in $I(S, x)$*

$$x \in I(S, x) = M_J(S)x + \hat{M}_J(S)\, U(x) S + \hat{M}_J(S)\, U(S)\, U(x) J \Rightarrow x = 0.$$

Proof. Suppose $x = M(x)$ as in (4.1). Our first step is to get rid of the operator M_0 using local nilpotence: $M_0 = M_0(t_1, \ldots, t_m)$ involves only a finite number of elements from S and has codegree $\geqslant 1$, and $\Phi[x, t_1, \ldots, t_m]$ is nilpotent by local nilpotence of S, hence $M_0{}^n x = 0$ for suitably large n. But then

$$x = Ix = (I + M_0 + \cdots + M_0^{n-1})(I - M_0)x = M_0'(x - M_0 x)$$
$$= \sum M_0' M_i U(x) s_i + \sum M_0' M_j U(s_j)\, U(x) a_j\,.$$

Thus we can rewrite x in the form (4.1) with $M_0 = 0$; assume this has been done,

$$x = \sum M_i U(x) s_i + \sum M_j U(s_j)\, U(x) a_j\,. \tag{4.1$'$}$$

But then, by (3.4), x pushes J into I_0, $U(x)J \subset I_0(S, x)$, and comparison of (3.2) and (3.3) shows $I(S, x) = I_0(S, x)$. Thus for such x we obtain $x \in I_0(S, x)$ and therefore $x = 0$ by (4.2). ∎

Our assumption that R was not solvable has led to a contradiction if we assume R (and therefore S) is locally nilpotent.

4.4. Theorem. *If J has* d.c.c. *on inner ideals contained in a locally nilpotent ideal R, then R is solvable.* ∎

Since local nilpotence is equivalent to local solvability [8, p. 480], this result says that a locally solvable ideal is globally solvable in the presence of the d.c.c. Our next step is to see that a radical R is always locally solvable.

5. Radical Implies Locally Nilpotent

We know [7, 4, 5] that in the presence of the d.c.c. on inner ideals a radical ideal spawns trivial elements : $R \neq 0$ iff $Z(R) \neq 0$ iff $Z_J(R) \neq 0$, for $Z(R)$ the ideal spanned by all elements of R trivial on R, and $Z_J(R) \subset R \cap Z(J)$ the ideal spanned by all elements of R trivial on J. (Namely, if B is a minimal nonzero inner ideal contained in R and $x \in B$ is nonzero then $x \notin U(x)J$ since no radical element is regular, therefore the inner ideal $U(x)J < B$ must be zero by minimality and x is trivial on J.)

Our effort will be to prove that $Z_J(R)$ is locally nilpotent. By radical surgery this is enough to prove all of R is locally nilpotent. Namely, because local nilpotence coincides with local solvability there exists a maximal locally nilpotent ideal, the *locally nilpotent radical* Loc(J), such that $\bar{J} = J/\mathrm{Loc}(J)$ has no locally nilpotent ideals. Proving R is locally nilpotent is the same as proving $R \subset \mathrm{Loc}(J)$, i.e., $\bar{R} = 0$ in $\bar{J}$ (note $\bar{J}$ still has d.c.c. on inner ideals inside $\bar{R}$). And to prove

$\bar{R} = 0$ we have seen it suffices to prove $Z_{\bar{J}}(\bar{R}) = 0$. Thus radical surgery allows us to replace J, R by $\bar{J}$, $\bar{R}$ and try to prove

$$\mathrm{Loc}(J) = 0 \Rightarrow Z_J(R) = 0 \text{ if } J \text{ has d.c.c. on inner ideals in } R, \tag{5.1}$$

or the equivalent version

$$Z_J(R) \text{ is locally nilpotent if } J \text{ has d.c.c. on inner ideals in } R. \tag{5.2}$$

Using combinatorial methods stemming from Shirshov, Slin'ko was able in 1972 [11] to show that $Z(J)$ is always locally nilpotent (without any finiteness conditions) in the case of *special* Jordan algebras. In 1977 Zelmanov [12] extended these results to nonspecial algebras with a weak d.c.c. In 1979 he settled a long outstanding conjecture by showing $Z(J)$ and $L(J)$ are *always* contained in $\mathrm{Loc}(J)$. This is a deep result, and here we will content ourselves with the easier version assuming the d.c.c.

The key to handling trivial elements, as Zelmanov recognized, is finite dimensionality. We call a unital module M over an arbitrary (commutative, associative) ring of scalars Φ *finite dimensional* if it has a.c.c. and d.c.c. on submodules (equivalently, has a composition series). If Φ is a field this is of course the usual notion. Remember [16, p. 22; 1] that this property is contagious:

it is hereditary: if M is finite dimensional so is any submodule N and any factor module $\bar{M} = M/N$; (5.3)

it is recoverable: if M/N and N are finite dimensional, so is M; (5.4)

it is additive: any finite sum of finite dimensional modules is finite dimensional. (5.5)

In general the d.c.c. alone does not imply finite dimensionality: witness the canonical counterexample

$$\mathbb{Z}_{p^\infty} = \varinjlim \mathbb{Z}_{p^n} = \bigcup p^{-n}\mathbb{Z} \subset \mathbb{Q}/\mathbb{Z}$$

has the d.c.c. on $\mathbb{Z}$-submodules but not the a.c.c. (5.6)

However, the d.c.c. does imply finite dimensionality for *unital* modules over *Artinian* rings [1; 16, Proposition 12, p. 71]

if M is a unital module over an Artinian ring Φ and has d.c.c. on submodules, then it is finite dimensional. (5.7)

In finite-dimensional situations nilpotency comes easy: radical ideals are always nilpotent, by the usual proof constructing a *maximal* nilpotent subalgebra plus some results on finitely generated algebras.

5.8. ALBERT–ZHEVLAKOV THEOREM [8, Theorem 1, p. 482]. *If J is Jordan algebra which is finite dimensional as a Φ-module, then the radical* Rad(J) *is nilpotent. In particular, any radical ideal is nilpotent.* ▮

The connection between trivial elements and finite dimensionality is given by

5.9. ZELMANOV'S FINITE-DIMENSIONALITY LEMMA. *If z, w are trivial elements of J and J has* d.c.c. *on inner ideals inside Φz and $U(z, w)J$ and $V(z, w)J$, then these are finite dimensional.*

Proof. Remember these C are trivial, consisting entirely of trivial elements: any $U(\alpha z) = \alpha^2 U(z)$ and any $U(U(z, w)a)$ or $U(V(z, w)a)$ vanish by (0.11) if $U(z) = U(w) = 0$. Thus *every submodule M of C is an inner ideal of J, $U(M)J \subset U(C)J = 0 \subset M$,* so *the* d.c.c. *on inner ideals becomes the* d.c.c. *on submodules* of C. There is no reason why Φ need be Artinian—we are making no assumptions about Φ at all (if Φ were a field this entire discussion would be unnecessary). However, Φz and $U(z, w)J$ and $V(z, w)J$ remain unital modules over the factor ring $\bar{\Phi} = \Phi/z^{\perp}$ for $z^{\perp} = \{\alpha \in \Phi \mid \alpha z = 0\}$, and $\bar{\Phi}$ is *Artinian*: the Φ-module homomorphism $\Phi \to \Phi z$ via $\alpha \to \alpha z$ has kernel $z^{\perp}$, hence $\bar{\Phi} \cong \Phi z$ as Φ-modules, and since we saw he latter C has d.c.c. on Φ-submodules so does the former, and the Φ-submodules of $\bar{\Phi}$ are precisely the ideals. Since the Φ-submodules and $\bar{\Phi}$-submodules of C coincide, C has d.c.c. on $\bar{\Phi}$-submodules and so by (5.7) it is finite dimensional over $\bar{\Phi}$ and Φ. ▮

Using radical surgery and finiteness, the result that radical implies locally nilpotent can be formulated in four equivalent ways.

5.10. ZELMANOV I. *If J has* d.c.c. *on all inner ideals inside a radical ideal R, then $R \subset$* Loc(J) *is locally nilpotent.*

5.11. ZELMANOV II. *If J has* d.c.c. *on inner ideals inside Φz and $U(z, w)J$ and $V(z, w)J$ for all trivial elements z, w in an ideal R, then $Z_J(R)$ is locally nilpotent.*

5.12. ZELMANOV III. *If J is generated by strictly trivial elements $\{z_i\}$ and has* d.c.c. *on inner ideals contained inside Φz_i and $U(z_i, z_j)\hat{J}$ and $V(z_i, z_j)\hat{J}$, then J is locally nilpotent and locally finite dimensional.*

5.13. ZELMANOV IV. *If J is generated by a finite number of strictly trivial elements $z_1, \ldots, z_n$ and has* d.c.c. *on inner ideals inside Φz_i and $U(z_i, z_j)\hat{J}$ and $V(z_i, z_j)\hat{J}$, then J is nilpotent and finite dimensional.*

In III and IV we are tacitly considering *nonunital Jordan algebras J*; every such can be embedded in its unital hull $\hat{J} = \Phi 1 + J$. When we are dealing with ideals R inside J, as in I and II, it does no harm to replace J by $\hat{J}$ and assume from the start that J is unital. However, when we are trying to prove nilpotence of J itself we clearly do not want to toss in a unit. In the nonunital case one must deal with *strictly trivial* elements, those that remain trivial in the unital hull (i.e., $z^2 = 0$ as well as $U(z)J = 0$).

A few words about the equivalence of these four versions. We saw in (5.2) that Zelmanov I follows from Zelmanov II. Zelmanov II follows from Zelmanov III applied to $Z_J(R)$: $Z_J(R)$ is spanned by the strictly trivial elements $\{z_i\}$ from R, and if J has d.c.c. on inner ideals inside Φz and $U(z, w)J$ and $V(z, w)J$ for trivial z, w from R (i.e., from $Z_J(R)$), then by 5.9 it has d.c.c. on all *submodules* inside these, hence on all submodules inside Φz and $U(z, w)\,\widehat{Z_J(R)}$ and $V(z, w)\,\widehat{Z_J(R)}$, so III is applicable to $Z_J(R)$. The global case III reduces to the local case IV for the usual reasons: any finitely generated subalgebra of J is contained in one of the form $J_0 = \Phi[z_1, \ldots, z_n]$, and if the latter is nilpotent and finite dimensional so is the original subalgebra. Thus we will concentrate our attention on IV.

The original proof [15] of IV detoured through Slin'ko's case of special algebras. However, an observation of Zelmanov [13] makes this unnecessary and leads immediately to finite dimensionality. The idea is to use the $V(z, w)$'s to capture the $V(z)$'s.

5.14. LEMMA. *If J is generated by elements $\{z_i\}$, then*

$$J = \sum \Phi z_i + \sum \{V(z_i, z_j)\,\hat{J} + U(z_i, z_j)\,\hat{J} + U(z_i)\hat{J}\} + \sum V(z_k)\,U(z_i, z_j)\hat{J}.$$

Proof. Denote the right side of the above expression by K. For any generating set we have $J = \sum \Phi z_i + M(J)\hat{J}$, where $M(J)$ is generated by the $V(z_i)$, $V(z_i, z_j)$, $U(z_i, z_j)$, $U(z_i)$. Thus each multiplication operator is a sum of monomial operators M in these generators. If M begins with $V(z_i, z_j)$, $U(z_i, z_j)$, or $U(z_i)$ then $M\hat{J} \subset K$ by definition. If it begins with $V(z_k)$ we have $M = V(z_k)M'$. If M' begins with $U(z_i, z_j)$ we have $V(z_k)\,U(z_i, z_j)\,M''\hat{J} \subset K$ by definition; if M' begins with $U(z_i)$ we can (by linearized (0.2)) replace M by monomials beginning with $U(z_i)$ or $U(z_i, z_k)$ or $V(z_i)\,U(z_i, z_k)$, whence $M\hat{J} \subset K$ by the above; if M' begins with $V(z_i)$ we note $V(z_k)\,V(z_i) = V(z_k, z_i) + U(z_k, z_i)$ by (0.5); if M' begins with $V(z_i, z_j) = V(z_i)\,V(z_j) - U(z_i, z_j)$ we apply the previous cases; and finally, if $M' = I$ then $M = V(z_k)$ has $M\hat{J} = V(z_k)(\Phi 1 + J) = 2\Phi z_k + V(z_k)(\sum \Phi z_i + M(J)\hat{J}) = 2\Phi z_k + \sum \Phi V(z_k, z_i)1 + V(z_k)\,M(J)\hat{J} \subset K$ by the previous cases $V(z_k)\,M'$ for $M' \neq I$. ∎

In particular, if J is generated by a finite number of elements $z_1, \ldots, z_n$ such that all Φz_i, $V(z_i, z_j)\hat{J}$, $U(z_i, z_j)\hat{J}$, $U(z_i)\hat{J}$ are finite dimensional, then J itself is finite dimensional by (5.5) (being a finite sum of finite-dimensional spaces, recalling that by (5.3) if C is finite dimensional so is the homomorphic image $V(z_k)C$).

This immediately establishes Zelmanov IV: the strictly trivial generators $z_1, \ldots, z_n$ have $U(z_i)\hat{J} = 0$ by strictness, and Φz_i, $U(z_i, z_j)\hat{J}$, $V(z_i, z_j)\hat{J}$ are finite dimensional by hypothesis, so by our above remark J itself is finite dimensional and therefore nilpotent by Albert–Zhevlakov 5.8. ▌▌▌▌

An unsettling possibility in the Jacobson structure theory was the existence of a simple algebra with d.c.c. which was radical (and without unit, therefore completely unknown). Once we have Zelmanov I we can rule out this possibility: the radical is locally nilpotent, and Slater–Zhevlakov 4.3 shows that a simple algebra cannot be locally nilpotent.

5.15. THEOREM. *A simple Jordan algebra with* d.c.c. *on inner ideals is semisimple, therefore nondegenerate with unit element as described in the structure theory.* ▌

Our next step, an easier one, is to pass from solvability to Penico-solvability.

6. SOLVABLE IMPLIES PENICO-SOLVABLE

Characteristic 2 presents a serious obstacle for the first time in the passage from solvability to Penico-solvability. We can begin smoothly with a reduction to the case of an ideal with $D(R) = 0$. Indeed, if R is solvable we have $R \supset D(R) \supset \cdots \supset D^n(R) = 0$ for some n, where each $D^k(R)$ is an ideal in J. Penico-solvability is recoverable: if R/S and S are Penico-solvable in J/S and J, respectively, then R is Penico-solvable in J ($P_J{}^n(\bar{R}) = 0$ and $P_J{}^m(S) = 0$ imply $P_J^{n+m}(R) \subset P_J{}^m(S) = 0$). Therefore it suffices to prove each $R_k = D^k(R)/D^{k+1}(R)$ is Penico-solvable in $J_k = J/D^{k+1}(R)$, where in J_k we still have the d.c.c. on inner ideals inside R_k but now have $D(R_k) = 0$. Thus replacing J, R by J_k, R_k makes our problem simpler: instead of solvability $D^n(R) = 0$ we may assume $D(R) = 0$.

In the same way we can further reduce to the case $P(R) = R$. Namely, if the Penico-derived sequence *did not* terminate it would stabilize by the d.c.c. at some nonzero Penico-idempotent ideal: $R \supset P(R) \supset \cdots \supset P^n(R) = P^{n+1}(R) = \cdots = S$, where J still has d.c.c. on inner ideals inside S and still $D(S) \subset D(R) = 0$, but now in addition $P(S) = S$. Replacing R by S, we may assume $D(R) = 0$ and $P(R) = R$ and try to prove $R = 0$.

For linear Jordan algebras over Artinian rings of scalars we do not need to stop at Penico-solvability: solvability implies nilpotence, by an old result of Morgan's.

6.1. MORGAN'S LEMMA. *Let J be a Jordan algebra over an Artinian ring Φ containing $\frac{1}{2}$. If J has* d.c.c. *on inner ideals insider an ideal R with $D(R) = 0$, then R is finite dimensional and nilpotent.*

Proof. By Albert–Zhevlakov 5.8 it suffices to prove finite dimensionality, and because we are dealing with unital modules over an *Artinian* ring Φ, (5.7) shows finite dimensionality is the same as the d.c.c. on submodules. Consider the chain $R \supset P(R) \supset R^2 \supset 0$. Here $R/P(R)$ and R^2 are trivial, so all subspaces are inner ideals: $R/P(R)$ is by definition, $U(R)J \subset P(R)$, and R^2 is since $U(R^2)J \subset D(R) = O\,(U(R^2)J$ is spanned by $U(r^2)J \subset U(r)R$ by (0.1), and by $U(r^2, s^2)J = \{U(r, s^2)\, V(r) - V(s^2)\, U(r)\}J$ (by (0.8)) $\subset U(R, R)R - V(R, R)R$ (by (0.4)) $\subset D(R))$. Therefore all their subspaces are inner ideals (in $J/P(R)$ or J), so the d.c.c. on inner ideals becomes the d.c.c. on subspaces and therefore finite dimensionality for these spaces. By (5.4) it suffices to prove $P(R)/R^2$ is finite dimensional.

Now for linear Jordan algebras the ideal $P(R)$ coincides with Penico's original formulation,

$$P(R) = R^2 + R^2 \circ J \qquad (\tfrac{1}{2} \in \Phi,\ R \lhd J),$$

since $P(R) \supset U(R)1 = R^2$ and hence always contains the right-hand side, while on the other hand, by (0.5), $2U(R)J = \{V(R)^2 - V(R^2)\}J$ shows $2P(R) \subset R \circ R + R^2 \circ J$. By finite dimensionality, R^2 is spanned by some $r_1{}^2, \ldots, r_n{}^2$. Then $P(R) = \sum \Phi r_i{}^2 + \sum r_i{}^2 \circ J$ is a finite sum of finite-dimensional spaces, and therefore itself finite dimensional by (5.5). (We saw above $\Phi r^2 \subset R^2$ is trivial, and each $r^2 \circ J$ is trivial since, by (0.12), $U(r^2) = 0$ implies $U(r^2 \circ J) = -U(U(J)\, r^2, r^2) = -V(r^2)\, V(U(J)\, r^2) + V(r^2, U(J)\, r^2)$ (by (0.5)) $= -V(r^2)\, V(r') + V(U(r^2)J, J)$ (by (0.3)) $= -V(r, r)\, V(r') + 0$ (by (0.3)) maps J into $V(R, R)R \subset D(R) = 0$; once these spaces are trivial they have d.c.c. on all subspaces, hence are finite dimensional). ∎

This depends heavily on Artinianness of Φ. Indeed, for non-Artinian scalars a whole new proof is necessary, because in fact R need *not* be finite dimensional (consider the case of the trivial algebra $R = Q_{p^{\infty}}$ over $\Phi = \mathbf{Z}$ as in (5.6)). Even in the Artinian case, when $\frac{1}{2} \notin \Phi$ we cannot get easily from $P(R)$ to $R^2 + R^2 \circ J$.

We try a different approach, using the d.c.c. on certain annihilator inner ideals.

6.2. Annihilator Lemma. *If X is a subset of a unital Jordan algebra J then*

$$\begin{aligned} \mathrm{Ann}_{U,J}(X) &:= \{z \in J \mid U(z)X = U(z, J)X = 0\} \lhd J, \\ \mathrm{Ann}_{V,J}(X) &:= \{z \in J \mid U(z)\, U(J)X = V(z, J)X = 0\} \lhd J \end{aligned} \tag{6.3}$$

are inner ideals of J. If $R \lhd J$ is an ideal then so are the annihilators

$$\mathrm{Ann}_J(R) = \mathrm{Ann}_U(R) = \mathrm{Ann}_V(R) \lhd J, \qquad \mathrm{Ann}(R) = R \cap \mathrm{Ann}_J(R) \lhd J \tag{6.4}$$

and for any subset $X \subset R$ we have inner ideals

$$\mathrm{Ann}_U(X) = R \cap \mathrm{Ann}_{U,J}(X), \qquad \mathrm{Ann}_V(X) = R \cap \mathrm{Ann}_{V,J}(X) \triangleleft J. \quad (6.5)$$

When $D(R) = 0$ these simplify to

$$\mathrm{Ann}_U(X) = \{z \in R \mid U(z, J)X = 0\}, \qquad \mathrm{Ann}_V(X) = \{z \in R \mid V(z, J)X = 0\}. \quad (6.6)$$

When both $D(R) = 0$ and $P(R) = R$ we have

$$\mathrm{Ann}(\bar{R}) = 0 \quad \text{in} \quad \bar{J} = J/\mathrm{Ann}(R) \quad \text{and} \quad \bar{R} = 0 \Leftrightarrow R = 0. \quad (6.7)$$

When $\frac{1}{2} \in \Phi$ these annihilators simplify to

$$\mathrm{Ann}_{U,J}(X) = \{z \mid U(z, J)X = 0\}, \qquad \mathrm{Ann}_{V,J}(X) = \{z \mid V(z, J)X = 0\},$$
$$\mathrm{Ann}_J(R) = \{z \mid (z \circ J) \circ R = 0\}, \qquad (\tfrac{1}{2} \in \Phi).$$

Proof. $\mathrm{Ann}_{U,J}(X)$ is clearly a linear subspace of J since $U(z, z') \in U(z, J)$; it is inner, $U(z)a \in \mathrm{Ann}_{U,J}(X)$ for $a \in J$, since, by (0.1), $U(U(z)a)X = U(z)U(a)\{U(z)X\} = 0$ and, by (0.8), $U(U(z)a, J)X = \{U(z, J)\, V(a, z) - V(J, a)U(z)\}X = 0$, noting $V(a, z) = V(a)\, U(z, 1) - U(z, a)$ by (0.5).

The defining condition for $\mathrm{Ann}_{V,J}(X)$ is, despite appearances, linear in z: for $z, z' \in \mathrm{Ann}_{V,J}(X)$ we have, by (0.9), $U(z, z')\, U(J)X \subset \{V(z, J)\, V(z', J) - V(z, U(J)\, z')\}X = 0$. Innerness follows from $U(U(z)a)\, U(J)X = U(z)\, U(a)\{U(z)\, U(J)X\} = 0$ from (0.1) and $V(U(z)a, J)X = \{V(z, a)\, V(z, J) - U(z)\, U(a, J)\}X = 0$ from (0.9).

For (6.4) we have equality of the two annihilators because $U(J)R = R$ for an ideal R in a unital J, because $V(z, J) = U(z, 1)\, V(J) - U(z, J)$ and $U(z, J) = V(z, 1)\, V(J) - V(z, J)$ by (0.5), and because $V(J)R \subset R$. To see $\mathrm{Ann}_J(R)$ is an ideal we need only check outerness $U(a)z \in \mathrm{Ann}_J(R)$: $U(U(a)z)R \subset U(a)\{U(z)R\} = 0$ by (0.1), $U(U(a)z, J)R = \{U(a, z)\, V(J, a) - V(J, z)\, U(a)\}R$ (by (0.8)) $\subset U(z, J)R - \{V(J)\, U(z, 1) - U(z, J)\}R = 0$ by (0.5) again.

(6.5) and (6.6) are clear. For (6.7), suppose $D(R) = 0, P(R) = R$, and $\bar{z} \in \mathrm{Ann}(\bar{R})$. Thus $z \in R$ has, by (6.6), $V(z, J)R \subset \mathrm{Ann}(R)$, so $V(R, J)\, V(z, J)R \subset V(R, J)\, \mathrm{Ann}(R) = V(\mathrm{Ann}(R), J)R = 0$. But then by P-idempotence and D-triviality of R, $V(R, J)z = V(P(R), J)z = V(U(R)J, J)z \subset V(R, J)\, V(R, J)z - U(R)R$ (by (0.9) since $z \in R$) $= V(R, J)\, V(z, J)R - 0 = 0$, so $V(z, J)R = 0$ and $z \in \mathrm{Ann}(R)$ by (6.6) and $\bar{z} = 0$. Further, if $\bar{R} = 0$ then $R \subset \mathrm{Ann}(R)$, $U(R, R)J = V(R, J)R = 0$. But this forces $R = 0$, since $U(R, R)J$ is all of R when $R = P(R) = P^2(R)$ and $D(R) = 0$:

$$P(R) \text{ is spanned by trivial elements, and}$$
$$P^2(R) = U(P(R), P(R))J \subset U(R, R)J \text{ when } D(R) = 0. \quad (6.8)$$

Indeed, $P(R)$ is spanned by the trivial elements $U(r)a$ when $D(R) = 0$ (using (0.1)), so $P^2(R) = U(P(R))J$ is spanned by the $U(U(r)a, U(r')\,a')J$.

Since $2U(z) = U(z, z)$ we see (via (0.9) in case of $\mathrm{Ann}_V(X)$) that the annihilators reduce as indicated for a set X. Using linearized (0.5) and (0.4) we obtain the reduction for an ideal R, $2\{zJR\} \subset z \circ (J \circ R) + R \circ (z \circ J) - (z \circ R) \circ J$ and $(z \circ J) \circ R \subset \{zJR\} + \{JzR\}$. ∎

6.9. *Remark.* There are a myriad of choices for annihilators which produce inner ideals of J, for example,

$$\mathrm{Ann}_M(X) = \{z \in J \mid U(x)z = U(x)\,U(z)J = U(x)\,U(z, J)J = 0 \text{ for all } x \in X\}$$
$$\mathrm{Ann}_N(X) = \{z \in J \mid V(J, z)X = U(J, J)\,U(z)X = 0\}.$$

We are led to these definitions if we wish to make the set of z with $U(z)X = 0$ or $U(z, J)X = 0$ (Ann_U), $V(z, J)X = 0$ (Ann_V), $U(X)z = 0$ (Ann_M), or $V(J, z)X = 0$ (Ann_N) into an inner ideal. Note that in our notation $\mathrm{Ann}(J)$ is just the extreme radical of J. $\mathrm{Ann}_V(X)$ is suitably orthogonal to X: if $B \subset X$, $C \subset \mathrm{Ann}_V(X)$ are inner ideals then $\{CJB\} = 0$ shows $B + C$ is again an inner ideal, with $P^n(B + C) = P^n(B) + P^n(C)$.

The annihilator used by Slin'ko [15, Lemma 6, p. 385] for linear Jordan algebras was

$$Z(R) = \{z \in J \mid z \circ R^2 = (z \circ R) \circ R = 0\}.$$

By (0.5), (0.4) this coincides with

$$Z(R) = \{z \in J \mid U(R)z = V(R, R)z = 0\}$$

but this does not generalize smoothly to quadratic algebras. ∎

The radical surgery of (6.7) shows it suffices to replace R by $\bar{R}$, and prove $R = 0$ under the hypothesis

$$D(R) = 0, \qquad P(R) = R, \qquad \mathrm{Ann}(R) = 0, \qquad J \text{ has d.c.c. inside } R.$$

This we do with the help of a well-chosen annihilator.

Using the d.c.c. on inner ideals contained in R, we can choose a minimal annihilator ideal of a *finite* set of nonzero *trivial* elements,

$$\mathrm{Ann}_V(X) = \{z \in R \mid V(z, J)X = 0\} \qquad (X = \{z_1, \ldots, z_m\},\ z_i \in R \text{ trivial}) \tag{6.10}$$

(recalling the simplification (6.6)). Now if x is any trivial element then $X' = X \cup \{x\} \supset X$ has annihilator $\mathrm{Ann}_V(X') \subset \mathrm{Ann}_V(X)$ of the same form, so by minimality we must have $\mathrm{Ann}_V(X') = \mathrm{Ann}_V(X)$: $z \in \mathrm{Ann}_V(X) \Rightarrow z \in \mathrm{Ann}_V(X') \Rightarrow z \in \mathrm{Ann}_V(x) \Rightarrow V(z, J)x = 0$ by (6.6). Since this is true for *any* trivial x, and R is spanned by such trivial elements by (6.8), we see $z \in \mathrm{Ann}_V(X) \Rightarrow V(z, J)R = 0 \Rightarrow z \in \mathrm{Ann}(R)$. But by hypothesis $\mathrm{Ann}(R) = 0$, so X is a *determining set*

$$z \in \mathrm{Ann}_V(X) \Rightarrow z = 0. \tag{6.11}$$

We use this determining set $X = \{z_1, \ldots, z_m\}$ to construct a "nonnilpotence sequence" $\{w_0, w_1, w_2, \ldots\}$ where the w_i all come from X (with lots of repetitions)) and for all n

$$W_n := V(w_n, J) \cdots V(w_1, J)\, w_0 \neq 0. \tag{6.12}$$

Indeed, we can choose $W_0 = w_0$ arbitrarily from X; if we have constructed $w_0, w_1, \ldots, w_n$ with $W_n \neq 0$ then, by (6.11) and (6.6), $W_n \neq 0 \Rightarrow W_n \not\subset \mathrm{Ann}_V(X) \Rightarrow V(W_n, J)X \neq 0 \Rightarrow V(X, J)\, W_n \neq 0 \Rightarrow W_{n+1} = V(w_{n+1}, J)\, W_n \neq 0$ for some $w_{n+1} \in X$.

Using this fixed sequence of trivial w's we define multilinear "independence-measuring functions" f_n: $J^n \to R$ via

$$f_n(a_n, \ldots, a_1) := V(w_n, a_n) \cdots V(w_1, a_1)\, w_0 \qquad (a_i \in J). \tag{6.13}$$

These are alternating functions,

$$f_n(\ldots a \ldots a \ldots) = 0, \qquad f_n(\ldots a \ldots b \ldots) = -f_n(\ldots b \ldots a \ldots) \tag{6.14}$$

since if adjacent variables coincide we have $f_n(\ldots aa \ldots) = 0$ by (0.9), $V(z, a)\, V(w, a) = V(z, U(a)w) - U(z, w)\, U(a) \in V(R, R) - U(R, R)\, U(J)$, where $V(R, R)$ and $U(R, R)\, U(J)$ map $V(w_{i-1}, a_{i-1}) \cdots V(w_1, a_1)\, w_0 \in R$ into $D(R) = 0$.

The kernel $K = \{a \in J \mid V(w_1, a)\, w_0 = U(w_1, w_0)a = 0\}$ has "finite codimension,"

$$J = \Phi b_1 + \cdots + \Phi b_r + K \tag{6.15}$$

since by Zelmanov finiteness 5.9 and the d.c.c. on inner ideals inside R, $U(w_1, w_0)J$ is finite dimensional, therefore spanned by a finite number of elements $U(w_1, w_0)\, b_i$, so any $a \in J$ has $U(w_1, w_0)a = \sum \beta_i U(w_1, w_0)\, b_i$ and $a - \sum \beta_i b_i \in K$.

K is not merely the kernel of f_1 : by alternativity (6.14), $f(J, \ldots, K, \ldots, J) = f_n(J, \ldots, J, K) = V(w_n, J) \cdots V(w_2, J)\, V(w_1, K)\, w_0 = 0$, so it lies in the kernel of any f_n,

$$f_n(J, \ldots, K, \ldots, J) = 0. \tag{6.16}$$

From (6.15) and (6.16) we see $f_n(J, \ldots, J)$ is spanned by the $f_n(b_{i_1}, \ldots, b_{i_n})$, so for $n > r$ at least one of the spanning elements $b_1, \ldots, b_r$ must appear twice and the whole function vanishes by alternativity (6.14),

$$f_{r+1}(J, \ldots, J) = 0.$$

But this is a *contradiction*, since $f_{r+1}(J, \ldots, J) = V(w_{r+1}, J) \cdots V(w_1, J)\, w_0 = W_{r+1} \neq 0$ by construction. Thus our original hypothesis that R was not Penico-solvable is untenable, and we have

6.17. THEOREM. *If J has* d.c.c. *on inner ideals contained in a solvable ideal R, then R is a Penico-solvable.* ▌

6.18. *Remark.* In the presence of a determining set (6.11) we can assume Φ is *Artinian*, and even a *field*, making the linear algebra more familiar. Indeed, whenever $X = \{z_1, \ldots, z_m\}$, where Φz_i are Artinian then

$$X^\perp = z_1{}^\perp \cap \cdots \cap z_m{}^\perp \qquad (z^\perp = \{\alpha \in \Phi \mid \alpha z = 0\})$$

is an ideal in Φ with $\bar{\Phi} = \Phi / X^\perp$ a finite subdirect sum of Artinian rings $\Phi_i = \Phi / z_i{}^\perp \cong \Phi z_i$,

$$\bar{\Phi} = \Phi / X^\perp \approx \Phi_1 \boxplus \cdots \boxplus \Phi_m \qquad (\Phi_i = \Phi / z_i{}^\perp \text{ Artinian}), \tag{6.19}$$

and therefore $\bar{\Phi}$ itself is Artinian. Now if $\alpha \in X^\perp$ then $\alpha X = 0$ implies

$$V(\alpha J, J)X = U(\alpha J, J)X = U(\alpha J)X = U(\alpha J)\, U(J)X = 0,$$

hence $\alpha J \subset \mathrm{Ann}_{V,J}(X) \cap \mathrm{Ann}_{U,J}(X)$,

$$X^\perp J \subset \mathrm{Ann}_{V,J}(X) \cap \mathrm{Ann}_{U,J}(X). \tag{6.20}$$

So far our argument has only used the fact that each Φz_i is Artinian. If we now use condition (6.11) we see $X^\perp R = 0$, i.e., $X^\perp = R^\perp$. Now it is enough to prove $\bar{R} = 0$ in the $\bar{\Phi}$-algebra $\bar{J} = J / R^\perp J$ since $R \subset R^\perp J \Rightarrow R = U(R, R)J \subset U(R, R^\perp J)J = 0$. Thus we may replace J, R, Φ by $\bar{J}$, $\bar{R}$, $\bar{\Phi}$ and assume Φ is Artinian.

The case of Artinian Φ easily reduces to the case of a field. It suffices to prove R is zero in the Φ/Ω-algebra $J/\Omega J$ for the nilpotent radical Ω of Φ, since if $R \subset \Omega J$ then $R = U(R, R)J = \Omega R$ and $R = \Omega^m R = 0$. Thus we may assume Φ is semisimple, hence a finite direct sum of fields. Since J and R break up into a direct sum of ideals which are algebras over these fields, it suffices to prove each summand of R is zero, i.e., to consider the case when Φ is a field. ▌

7. Penico-Solvable Implies Nilpotent

The final step in our analysis is to prove that Penico-solvable ideals R are nilpotent in the presence of the d.c.c. Here we will not be able to reduce to the case $D(R) = 0$ or $P(R) = 0$, and R will not be spanned by trivial elements. But we do not need trivial elements to provide finiteness as in Zelmanov finiteness, because *R now has the* d.c.c. *on all submodules*: in the chain $R \supset P(R) \supset \cdots \supset P^n(R) = 0$ all submodules B between $P^i(R)$ and $P^{i+1}(R)$ are inner ideals in J, $U(B)J \subset U(P^i(R))J = P^{i+1}(R)$, so that the d.c.c. on inner ideals contained in R shows each $P^i(R)/P^{i+1}(R)$ has d.c.c. on submodules and therefore R does too.

When Φ is a field, or more generally an Artinian ring, there is nothing to prove: the d.c.c. on submodules guarantees R is finite dimensional, so R is nilpotent by Albert–Zhevlakov 5.8. Our effort for general Φ will be to prove R is almost finite dimensional. The example (5.6) shows R itself need not be finite dimensional, but we can show it is finite dimensional modulo a suitably trivial ideal.

Once more we use a minimal annihilator, this time one of the form $\mathrm{Ann}_U(X)$ as in (6.3). Choose a minimal inner ideal among those of the form $\mathrm{Ann}_U(X)$ for a finite set $X = \{x_1, ..., x_n\}$ of (arbitrary) elements of R. As in (6.19) we have $\bar{\Phi} = \Phi/X^\perp$ Artinian since now any $\Phi/x^\perp \cong \Phi x$ is Artinian, and as in (6.20) $X^\perp R \subset \mathrm{Ann}_U(X)$. We no longer know this is zero as in (6.11), but it is still very trivial:

$$\mathrm{Ann}_U(X) = \mathrm{Ann}(R) = \{z \in R \mid U(z)R = U(z, J)R = 0\}. \tag{7.1}$$

Indeed, as before for any element $x \in R$ the set $X' = X \cup \{x\}$ remains finite, so $\mathrm{Ann}_U(X') \subset \mathrm{Ann}_U(X)$ forces $\mathrm{Ann}_U(X') = \mathrm{Ann}_U(X)$ by minimality, therefore $z \in \mathrm{Ann}_U(X) \Rightarrow z \in \mathrm{Ann}_U(X') \Rightarrow z \in \mathrm{Ann}_U(x)$ for any x in R, i.e., $z \in \mathrm{Ann}_U(R) = \mathrm{Ann}(R)$ by (6.4). In particular, $X^\perp R \subset \mathrm{Ann}_U(X)$ becomes

$$X^\perp R \subset \mathrm{Ann}(R)\colon U(X^\perp R)R = \{(X^\perp R)RJ\} = 0. \tag{7.2}$$

This almost shows R is nilpotent. Namely, $\bar{J} = J/X^\perp R$ has d.c.c. on inner ideals (even submodules) inside the Penico-solvable ideal $\bar{R} = R/X^\perp R$, which is an algebra over the Artinian ring $\bar{\Phi} = \Phi/X^\perp$, so as we observed above $\bar{R}$ is nilpotent and finite dimensional. If $\bar{R} = \bar{\Phi}\bar{r}_1 + \cdots + \bar{\Phi}\bar{r}_m$ and $\bar{R}^n = 0$ then back inside R we have

$$R = \Phi r_1 + \cdots + \Phi r_m + X^\perp R, \qquad R^n \subset X^\perp R. \tag{7.3}$$

(Note that $2R^{n+2} \subset 2\{U(R) + V(R)\}\, R^n \subset \{U(R, R) + V(R)\}\, X^\perp R = 0$ by (7.3), (7.2), so for linear Jordan algebras where $\frac{1}{2} \in \Phi$ we can stop here: the following arguments are necessary only when Φ is not Artinian and $\frac{1}{2} \in \Phi$.)

We have

$$R^{2p+2n} \subset U(R)^p(X^{\perp}R) \tag{7.4}$$

since $R^{k+2} \subset M_k(R)R$ and $M_{2k}(R) \subset M(R)^k$ [8, (22), (23), p. 474] show $R^{2p+2n} = R^{2(p+n-1)+2} \subset M(R)^{p+n-1}R \subset M(R)^p\, R^n \subset M(R)^p\, X^{\perp}R$ (by (7.3)), and the only operators from $M(R)$ which act nontrivially are the $U(R)$ since by (7.2)

$$U(R, R)\, X^{\perp}R = V(R)\, X^{\perp}R = 0.$$

We define another "independence-measuring function"

$$f_k\colon R \times \cdots \times R \to \operatorname{End}(R)/I(R) \text{ for } I(R) = \{T \in \operatorname{End}(R) \mid T(X^{\perp}R) = 0\}$$
$$\text{via } f_k(x_1, \ldots, x_k) = U(x_1) \cdots U(x_k). \tag{7.5}$$

Here $I(R)$ is an "ideal" invariant under multiplication by $M(J)$, and by (7.2) we see

$$I(R) \text{ contains } V(R),\ U(R, R),\ U(X^{\perp}R),\ V(J, R),\ V(R, J) \tag{7.6}$$

(noting $V(R, J) \subset V(J, R)$ by (0.4)). These f_k are "multilinear,"

$$U(\alpha x) = \alpha^2 U(x), \qquad U(x + y) \equiv U(x) + U(y) \qquad \operatorname{mod} I(R) \tag{7.7}$$

since $U(x, y) \in I(R)$, they vanish on $X^{\perp}R$ by (7.6),

$$U(X^{\perp}R) \equiv 0 \qquad \operatorname{mod} I(R), \tag{7.8}$$

and they are "alternating modulo $P(R)$,"

$$\begin{gathered} f_k(\ldots, x, \ldots, x, \ldots) \in \sum f_{k-1}(R, \ldots, P(R), \ldots, R) \\ U(x)\, U(x) = U(x^2),\ U(x)\, U(y) - U(y)\, U(x) \equiv U(x \circ y) \qquad \operatorname{mod} I(R) \end{gathered} \tag{7.9}$$

using (0.1), (0.13) where x^2, $x \circ y \in P(R)$.

Since R is spanned by $r_1, \ldots, r_m$ modulo $X^{\perp}R$ by (7.3), and the "alternating multilinear functions" f_k vanish on $X^{\perp}R$ by (7.8), $f_k(R, \ldots, R)$ is spanned by the $f_k(r_{i_1}, \ldots, r_{i_k})$. For $k > m$ at least one spanning element r_i must appear twice, hence by the alternating nature (7.9) f_k must fall into $P(R)$

$$f_{m+1}(R, \ldots, R) \subset \sum f_m(R, \ldots, P(R), \ldots, R). \tag{7.10}$$

We have now gathered enough tools to prove

7.11. Theorem. *If J has* d.c.c. *on inner ideals contained in a Penico-solvable ideal R, then R is nilpotent.*

Proof. We induct on the index of Penico-solvability. Index 0 ($R = 0$) and index 1 ($P(R) = 0 \Rightarrow R^2 = 0$) are trivial. Assume the result for lesser indices. Since $P(R)$ has index one less than R, and J still has the d.c.c. on inner ideals inside $P(R)$, by induction $P(R)^N = 0$ for suitable N.

If R is spanned by m elements modulo $R \cap X^{\perp}J$ as in (7.3), we claim

$$f_{N(m+1)}(R,..., R) = 0. \tag{7.12}$$

Indeed, $f_{N(m+1)}(R,..., R) = U(R)^{N(m+1)} = \{U(R)^{m+1}\}^N \subset f_m(R,..., P(R),..., R)^N$ by (7.10). By (7.9) we can move the N factors $U(P(R))$ to the left (possibly reducing the number of factors $U(R)$, but keeping the same number of factors $U(P(R))$: $U(r)\, U(p) = -U(p)\, U(r) + U(p')$. Thus $f_{N(m+1)}(R,..., R) \subset U(P(R))^N \hat{M}(R)$, where the latter is actually zero as a map on R (not merely congruent to zero mod $I(R)$): it maps R into $U(P(R))^N \hat{M}(R) \subset U(P(R))^{N-1}P(R) \subset P(R)^{2N-1} \subset P(R)^N = 0$ by the inductive nilpotence of $P(R)$. Thus (7.12) holds.

From (7.12) we have $U(R)^{N(m+1)} \equiv 0 \bmod I(R)$, i.e., it falls in $I(R)$ and annihilates $X^{\perp}R$, so (7.4) shows

$$R^{2N(m+1)+2n} = 0$$

and R is nilpotent. This completes our induction. ∎

In view of Theorem 5.10, 4.4, 6.17, and 7.11 we have our main theorem.

7.13. ZELMANOV NILPOTENCE THEOREM. *If J is a Jordan algebra with* d.c.c. *on inner ideals contained in the radical ideal R, then R is nilpotent.* ∎

7.14. THEOREM. *If J has* d.c.c. *on inner ideals contained in the Jacobson radical* Rad(J), *then* Rad(J) *is nilpotent.* ∎

7.15. THEOREM. *A semiprime Jordan algebra with* d.c.c. *has unit and is a direct sum of simple nondegenerate ideals.* ∎

It is disappointing that this theorem has such a tortuous proof, whose key is the property $x \notin I(S, x)$ of locally nilpotent ideals S, instead of the short associative proof in Section 1 which makes clear and explicit use of quasi-invertibility $x \notin Sx$. It is possible that a more direct proof can be found (at least for linear Jordan algebras), on the other hand it may just be the price one must pay for the quadratic products involved in inner ideals and quadratic Jordan algebras.

Whatever its esthetic shortcomings, the proof does complete the gap in Jacobson's elegant structure theory for Jordan rings with d.c.c. on inner ideals, a theory revealing deep analogies with the Artin–Wedderburn theory of associative rings with d.c.c. on one-sided ideals.

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